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THE STABILITY OF THE ROTATION OF A HEAVY BODY WITH A VISCOUS FILLING†

M. Z. DOSAYEV and V. A. SAMSONOV
Moscow
e-mail: samson@inmech.msu.ru

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The stability of the permanent rotation of a symmetrical heavy body with a viscous filling is investigated. A finite-dimensional phenomenological model of the "internal friction" with which the filling acts on the wall of the cavity is constructed based on the Helmholtz equations for a vortex. The boundaries of the stability limit are constructed and the interaction between the internal friction and the external damping is tracked. It is shown that the cases of a cavity that is oblate and prolate along the axis of rotation lead to the existence of different forms of stability regions. © 2002 Elsevier Science Ltd. All rights reserved.

The modern formulation of the problem of the stability of the permanent rotation of a body with an ideal fluid about a vertical axis of symmetry was presented by Rumyantsev [1]. The case of the uniform vortex motion of a fluid was considered in [2]. Different versions of this problem have been discussed in [3] and elsewhere. A boundary layer was "added on" to the uniform vortex flow of the fluid and an estimate of the effect of the viscosity of the filling was given in [4].‡ The Helmholtz equations for a vortex were used§ when constructing a finite-dimensional phenomenological model of "internal friction".

Below, using this model, we investigate the nature of the stability of the permanent rotation of a body with a viscous filling.

1. FORMULATION OF THE PROBLEM

Consider the motion about a fixed point O of an axisymmetric heavy body, having an axisymmetric cavity completely filled with a viscous fluid. Suppose Oz is the axis of symmetry of the cavity and the vessel. For simplicity we will assume that the centre of the cavity coincides with the point C – the centre of gravity of the body-vessel.

We will connect with the body a system of coordinates $Oxyz$, directing axes along the principal axes of inertia of the body. We will introduce [5] a system of coordinates $OXYZ$, rotating about the vertical OZ. The orientation of the body is defined by three angles: γ the cyclic rotation of the OZx plane, which coincides with OZX plane, about the vertical) and α and β (the Krylov angles in the $OXYZ$ axes).

In addition to the gravity force $M\mathbf{g}$, we will take into account the external damping moment $\mathbf{D}$ (of an aerodynamic nature) and the interaction between the vessel walls and the filling, which is reduced to a resultant pair of forces with moment $\mathbf{L}$.

We will describe the motion of the system. The vector $\mathbf{G}_t$ of the angular momentum body about the fixed point O is related to the angular velocity of the body by the well-known relation $\mathbf{G}_t = \Theta_t\omega$, where the tensor Θ_t is formed by the principal moments of inertia $\tilde{A}_t, \tilde{A}_t, \tilde{C}_t$ of the body for the point O.

We will assume that the fluid filler performs so-called simple motion in the cavity [6]. Its state is then described by three components of the vortex $\mathbf{\Omega}$, and the vector of the angular momentum of the filler about the fixed point $\mathbf{G}_g$ can be represented in the form

$$\mathbf{G}_g = \Theta_0^*\boldsymbol{\omega} + \Theta'\mathbf{\Omega}$$

†*Prikl. Mat. Mekh.* Vol. 66, No. 3, pp. 427–433, 2002.

‡See also: SAMSONOV, V. A. and FILIPPOV, V. V., An experimental estimate of the effect of a fluid in the cavity of a body on its rotation. Report of the Institute of Mechanics, Moscow State University, No. 2386, 1980.

§SAVCHENKO, A. Ya., SAMSONOV, V. A. and SUDAKOV, S. N., A phenomenological model of interaction of a filler with the wall of a processing vessel. Report of the Institute of Mechanics, Moscow State Universty, No. 3617, 1988.

where the tensor Θ_0^* is formed by the moments of inertia $\tilde{A}^*$, $\tilde{C}^*$ of the so-called equivalent body, while the tensor Θ' is formed by the differences of the moments of inertia of the fluid and the equivalent body.

We will write the equation of motion of the system in the principal axes using the theorem of the change in the angular momentum about a fixed point,

$$(\Theta_t+\Theta^*)\frac{d\boldsymbol{\omega}}{dt}+\Theta'\frac{d\boldsymbol{\Omega}}{dt}+\boldsymbol{\omega}\times\left[(\Theta_t+\Theta^*)\boldsymbol{\omega}+\Theta'\boldsymbol{\Omega}\right]=\mathbf{M}_{\text{ext}} \tag{1.1}$$

where $\mathbf{M}_{\text{ext}}$ is the moment of the external forces acting on the system.

2. A MODEL OF THE "INTERNAL FRICTION"

We will assume that there are two mechanisms by which the filler and the rigid body interact: the normal pressure of the filler on the vessel wall $\mathbf{L}_p$ and the tangential friction of the fluid against the vessel wall $\mathbf{L}_f$. To take into account the normal pressure on the vessel wall we will transform the well-known Helmholtz equations for the uniform vortex flow of the fluid

$$\begin{aligned}
\dot{\Omega}_1&=\omega_3\Omega_2-(1-e)\omega_2\Omega_3-e\Omega_2\Omega_3=\delta_1\\
\dot{\Omega}_2&=-\omega_3\Omega_1+(1-e)\omega_1\Omega_3+e\Omega_1\Omega_3=\delta_2\\
\dot{\Omega}_3&=(1+e)(\omega_2\Omega_1-\omega_1\Omega_2)=\delta_3
\end{aligned} \tag{2.1}$$

where the parameter e represents the relative elongation of the cavity, together with Eq. (1.1), to the form

$$\dot{\mathbf{G}}_t+\boldsymbol{\omega}\times\mathbf{G}_t=\boldsymbol{\xi}+\mathbf{M}_{\text{ext}}=\mathbf{L}_p+\mathbf{M}_{\text{ext}},\quad \dot{\mathbf{G}}_g+\boldsymbol{\omega}\times\mathbf{G}_g=-\boldsymbol{\xi}=-\mathbf{L}_p \tag{2.2}$$

Here

$$\xi_1=L_{p1}=\frac{\tilde{A}_t}{A_t+A^*}\{A'\omega_3\Omega_2-C'\omega_2\Omega_3-A'\delta_1+(B^*-C^*+B_t-C_t)\omega_2\omega_3\}+$$

$$+\;(C_t-A_t)\omega_2\omega_3+\left(\frac{\tilde{A}_t}{A_t+A^*}-1\right)M_{\text{ext}1}\quad (1\ 2\ 3)$$

$$\tilde{A}_t=A_t+M_t s^2,\quad B_i=A_i$$

M_t is the mass of the body and s is a variable parameter – the coordinate of the point C on the Oz axis

To introduce the friction of the filling against the vessel wall, we will add the moment $\mathbf{L}_f$ to the right-hand side of the first equation of (1.3) and the moment $-\mathbf{L}_f$ to the right-hand side of the second equation, respectively. We obtain the following system

$$\dot{\mathbf{G}}_t+\boldsymbol{\omega}\times\mathbf{G}_t=\mathbf{L}_p+\mathbf{L}_f+\mathbf{M}_{\text{ext}} \tag{2.3}$$

$$\dot{\mathbf{G}}_g+\boldsymbol{\omega}\times\mathbf{G}_g=-\mathbf{L}_p-\mathbf{L}_f \tag{2.4}$$

We will assume that the moment $\mathbf{L}_f$ depends linearly on the difference of the vortex vector of the filler and the angular velocity of the body

$$\mathbf{L}_f=\sigma(\boldsymbol{\Omega}-\boldsymbol{\omega})$$

where σ is the coefficient of internal friction.

3. THE DYNAMICAL SYSTEM

We will replace system (2.3), (2.4) by the equivalent system 2.3, (2.1) and we will write the system obtained in the form

$$
\begin{aligned}
&A_*\dot\omega_1 + A'\dot\Omega_1 + (C_* - A_*)\omega_2\omega_3 + C'\omega_2\Omega_3 - A'\omega_3\Omega_2 = M_{\text{ext}1}\\
&A_*\dot\omega_2 + A'\dot\Omega_2 + (A_* - C_*)\omega_3\omega_1 + A'\omega_3\Omega_1 - C'\omega_1\Omega_3 = M_{\text{ext}2}\\
&C_*\dot\omega_3 + C'\dot\Omega_3 + A'(\omega_1\Omega_2 - \omega_2\Omega_1) = M_{\text{ext}3}\\
&\dot\Omega_1 = \omega_3\Omega_2 - (1-e)\omega_2\Omega_3 - e\Omega_2\Omega_3 - \frac{\sigma A_*}{A'\bar A_t}(\Omega_1 - \omega_1)\\
&\dot\Omega_2 = -\omega_3\Omega_1 + (1-e)\omega_1\Omega_3 + e\Omega_1\Omega_3 - \frac{\sigma A_*}{A'\bar A_t}(\Omega_2 - \omega_2)\\
&\dot\Omega_3 = (1+e)(\omega_2\Omega_1 - \omega_1\Omega_2) - \frac{\sigma C_*}{C'C_t}(\Omega_3 - \omega_3)\\
&\omega_1 = \dot\alpha - \dot\gamma\sin\beta,\quad \omega_2 = \dot\beta\cos\alpha + \dot\gamma\cos\beta\sin\alpha,\quad \omega_3 = -\dot\beta\sin\alpha + \dot\gamma\cos\beta\cos\alpha
\end{aligned}
\tag{3.1}
$$

We will determine the moment **D** of the internal damping forces. It is usual to split this into two parts. The first, the so-called "quenching" moment, leads to a reduction in the axial component of the angular velocity of the body (this part is not considered here). The second part ensures dissipation of only the angular motions of the axis of the body and can be represented in the form

$$\mathbf{D} = (D_X, D_Y, D_Z) = (-A_*k_1(\dot\alpha - \omega_0\beta), -A_*k_1(\dot\beta + \omega_0\alpha), 0)$$

where k_1 is the external damping coefficient.

This also completes the construction of the ninth-order dynamical system with nine phase variables: Ω_i $(i = 1,2,3)$, $\alpha, \beta, \gamma, \dot\alpha, \dot\beta, \dot\gamma$.

This system has an obvious solution, corresponding to the permanent rotation of the body and the filler about the vertical with angular velocity ω_0

$$\alpha = \beta = 0,\quad \dot\alpha = \dot\beta = \Omega_1 = \Omega_2 = 0,\quad \dot\gamma = \Omega_3 = \omega_0 \tag{3.2}$$

When $s > 0$, we have the upper equilibrium position, and when $s > 0$, we have the lower equilibrium position.

4. THE STABILITY OF PERMANENT ROTATION

To investigate the stability of solution (3.2) in the first approximation, we will linearize system (3.1). the linearized system is then split into two independent subsystems. One of these is a sixth-order subsystem in the variables α, β, Ω_1, Ω_2

$$
\begin{aligned}
\ddot\alpha &= \alpha\left\{\frac{Mgs}{A_*} + \frac{(A'(1-e) - C')}{A_*}\omega_0^2 + \frac{(A_* - C_*)}{A_*}\omega_0^2\right\} + \beta\frac{\sigma\omega_0}{\bar A_t} - \beta\omega_0 k_1 - \dot\alpha\frac{\sigma}{\bar A_t} -\\
&- \dot\alpha k_1 + \dot\beta\left(\frac{A'(1-e) - C'}{A_*} + \frac{A_* - C_*}{A_*} + 1\right)\omega_0 + \Omega_1\frac{\sigma}{\bar A_t} + \Omega_2\frac{A'e\omega_0}{A_*}\\
\ddot\beta &= \beta\left\{\frac{Mgs}{A_*} + \frac{A'(1-e) - C'}{A_*}\omega_0^2 + \frac{A_* - C_*}{A_*}\omega_0^2\right\} - \alpha\frac{\sigma\omega_0}{\bar A_t} - \alpha k_1\omega_0 - \dot\beta\frac{\sigma}{\bar A_t} -\\
&- \dot\beta k_1 + \dot\alpha\left(\frac{C' - A'(1-e)}{A_*} + \frac{C_* - A_*}{A_*} - 1\right)\omega_0 - \Omega_1\frac{A'e\omega_0}{A_*} + \Omega_2\frac{\sigma}{\bar A_t}\\
\dot\Omega_1 &= -(1-e)\omega_0(\dot\beta + \alpha\omega_0) + (1-e)\omega_0\Omega_2 - \frac{\sigma A_*}{A'\bar A_t}(\Omega_1 - \dot\alpha + \beta\omega_0)
\end{aligned}
\tag{4.1}
$$

Here $M = M_i + M_g$ is the mass of the body-filler system.

The following first integral

$$C'\Omega_3 + C_t\dot\gamma = \text{const}$$

and the equation

$$(\Omega_3 - \dot{\gamma})^{\cdot} = -\sigma\left(\frac{1}{C'} + \frac{1}{C_t}\right)(\Omega_3 - \dot{\gamma})$$

can be obtained from the remaining second-order subsystem in the variables Ω_3, $\dot{\gamma}$.

A characteristic polynomial was extract from subsystem (4.1). The stability region was determined using well-known Hurwitz criterion. The Hurwitz conditions are extremely complex and cannot be investigated analytically. We therefore constructed the boundaries of the stability region numerically.

We will indicate the set of parameters of the problem for the numerical calculation. The description of the body requires three parameters: M_t, A_t and C_t. As previously,† we will assume that, to describe the inertial characteristics of the filler in the axisymmetric cavity, three parameters are sufficient, for example, M_g, a_1, and a_3 (a_1 and a_3 are the linear dimensions of the cavity). The remaining parameters are then found from the relations

$$e = \frac{a_3^2 - a_1^2}{a_3^2 + a_1^2}, \quad A' = \frac{4}{5} M_g \frac{a_1^2 a_3^2}{a_1^2 + a_3^2}, \quad A^* = \frac{M_g}{5} \frac{(a_1^2 - a_3^2)^2}{a_1^2 + a_3^2}, \text{ etc.}$$

We considered two types of body (1) $A_t < C_t$ (an oblate body) and (2) $A_t > C_t$ (a prolate body), for each of which we introduce two types of cavity (a): $a_1 < a_3$ (a prolate cavity) and (b) $a_1 > a_3$ (an oblate cavity).

In addition to these parameters it is also necessary to specify four others: σ, ω_0, s and k_1.

The boundaries of the stability region were projected onto the (s, ω_0) plane.

5. A TEST EXAMPLE

We will first consider the case $k_1 = 0$, when there is no external damping. In this case, the equations of motion have an area integral

$$G_Z = -G_x \sin\beta + G_y \cos\beta \sin\alpha + G_z \cos\beta \cos\alpha = \text{const}$$

(G_z is the projection of the angular momentum vector of the system onto the vertical) and, as is well known [6], the problem of the stability of permanent rotation (3.2) reduces to the problem of the nature of the extremum of the changed potential energy

$$W = G_Z^2 / 2J + MgZ_c$$

Where J is the moment of inertia of the system about the vertical.

It can be shown that the sufficient condition for a minimum of W and, consequently, for the stability of the rotation is the inequality

$$(C_t + C_g - A_t - A_g - Ms^2)\omega^2 - Mgs > 0 \tag{5.1}$$

This same inequality, but with the opposite sign, turns out to be the sufficient condition for instability of the rotation when $\sigma \neq 0$.

The stability condition (5.1) defines a stability region in parameter space.

We will not concern ourselves here with analysing the nature of the stability on the boundary of the stability region itself.

It is clear that $k_1 = 0$ for any σ, the boundaries of the stability region will be unchanged. Consider an oblate body. In this case there is a critical value of the parameter s

$$s_{cr}^2 = (C_t + C_g - A_t - A_g)/M$$

such that the lower equilibrium position when $s > -s_{cr}$ will always be stable, while the upper equilibrium position when $s < s_{cr}$ will be stable for fairly large ω_0 (Fig. 1a). Note that for a prolate body the upper equilibrium position $k_1 = 0$ is unstable for any ω_0.

Consider the case of an oblate body with a prolate cavity. We recall that inequality (5.1) does not contain the quantity σ – the coefficient of internal friction. Hence, the case considered was used for a

†SAMSONOV, V. A. and DOSAYEV, M. Z., A model of the motion of a top with a viscous filler on a rough plane. Report of the Institute of Mechanics, Moscow State University, No. 4485, 1997.

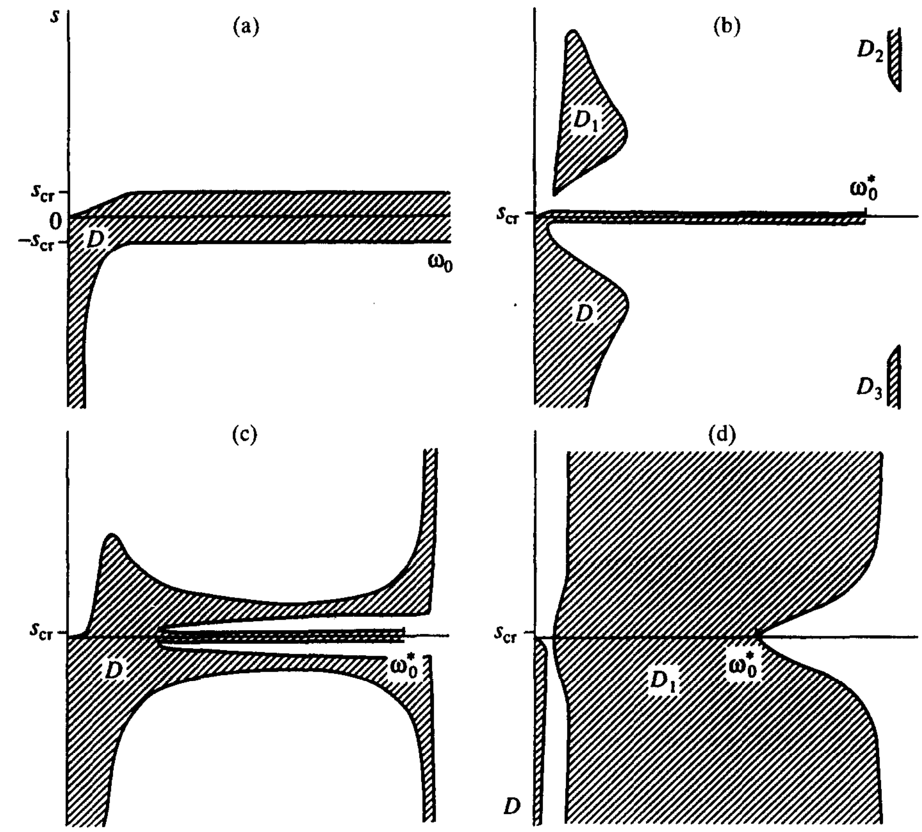

Fig. 1

test check of the numerical algorithms. The form of the stability region D obtained in the calculations for $k_1 = 0$ and any σ corresponds to the expected form.

To fix our ideas, suppose further that $\sigma = 4$ kgm^2/s. We will describe how the stability region changes when k_1 increases.

For fairly small $0 < k_1 \leqslant 10^{-3}$ s^{-1} the stability region D becomes bounded with respect to ω_0. Simultaneously, the "tongue" of stability increases for $s \ll -s_{cr}$.

When $k_1 \sim 10^{-2}$ s^{-1}, two bounded stability regions D_1 and D_2 appear in the half-plane $s > 0$. At the same time, a stability region D_3 occurs in the lower half-plane (Fig. 1b). The regions D_1, D_2 and D_3 combine with the region D and k_1 increases.

Further, as k_1 increases ($k_1 \sim 1$ s^{-1}) the stability region becomes simply-connected (Fig. 1c), and the inverse image of the region D_3 increases in the negative direction of the ordinate axis to infinity. The inverse image of the region D_2, in turn, increases in the positive direction.

As k_1 increases further ($k_1 \approx 700$ s^{-1}) the stability region increases and becomes doubly-connected (it splits into region D and D_1). The region D_1 in this case is shifted to the right (Fig. 1d). Further, when k_1 increases to values of the order of 10^5 s^{-1} the stability pattern changes considerably.

We will now discuss the effect of the parameter σ on the nature of the variation of the stability region as k_1 increases.

An increase in σ leads to "stretching" of the stability region. For example, for $k_1 = 0.01$ s^{-1}, as σ increases from 4 kgm^2/s to 40 kgm^2/s the size of the region D_1 (Fig. 1b) almost doubles, and when σ is reduced to 2 kgm^2/s the size of the region D_1 is reduced by a factor of 1.4. In addition to this, an increase in σ leads to "retardation" of the change in the stability region. For example, for $k_1 = 700$ s^{-1} with $\sigma = 4$ kgm^2/s we obtain a doubly-connected stability region (Fig. 1d). while for $\sigma = 6$ kgm^2/s we obtain a simply-connected region.

We will now consider the case of an oblate body with an ablate cavity. For $k_1 = 0$ and any σ we obtain a stability region similar to that for a prolate cavity (Fig. 1a). Suppose once again, to fix our ideas, that $\sigma = 4$ kgm^2/s. When $0 < k_1 \leqslant 10^{-4}$ s^{-1} the form of the stability region does not change appreciably. Unlike the case of a prolate cavity, the stability region D is not limited with respect to ω_0. When $k_1 \approx 10^{-3}$ s^{-1}, a bounded stability region D_1 appears in the upper half-plane (Fig. 2a), which combines

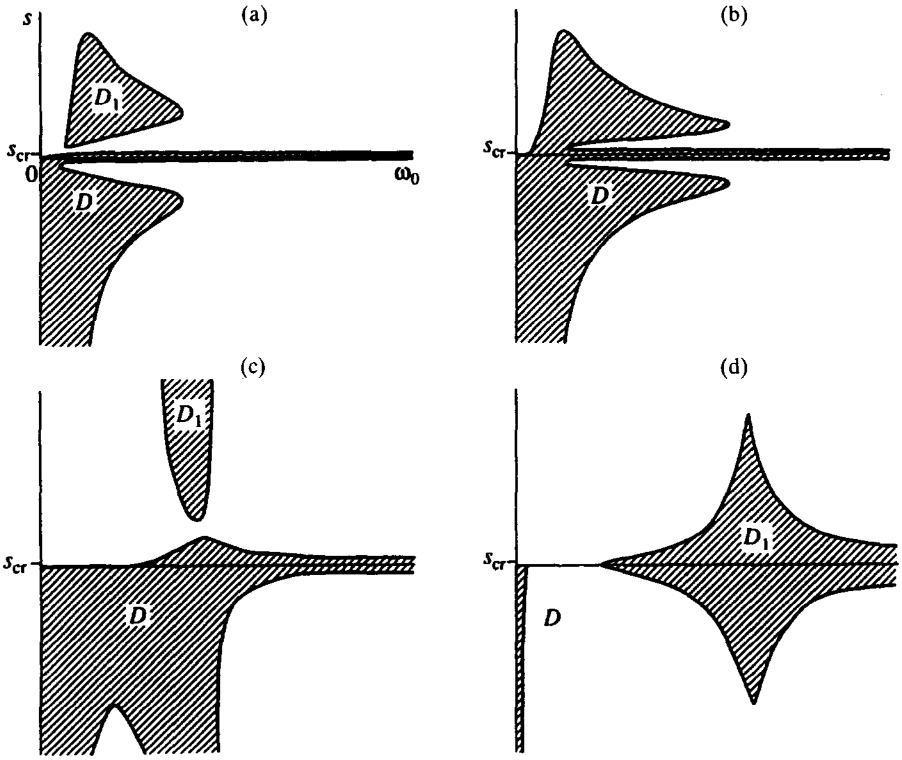

Fig. 2

with the region D as k_1 increases to $10^{-2}\ s^{-1}$ (Fig. 2b). As k_1 increases further, the "tongues" of stability expand and converge, and then combine with the central part of the stability region. When $k_1 \sim 40\ s^{-1}$ a "tooth" of instability appears in the lower half-plane, while in the upper half-plane a "tongue" of stability D_1 appears (Fig. 2c).

When $k_1 \sim 60\ s^{-1}$ the stability region becomes simply-connected. Later it once again splits into two. When k_1 increases further the "tongues" of stability in both half-planes become bounded and contract (Fig. 2d).

As in the case of a prolate cavity, an increase in the parameter σ leads to "stretching" of the stability region and to a slowing down in the change in the form of the stability region as k_1 increases.

It seems obvious that the case of a prolate body should differ qualitatively from the case of an oblate body only for $|s| < s_{cr}$. In fact, the results of a numerical calculation show that the stability pattern for the case of a prolate body is obtained by "cutting out" a strip $|s| < s_{cr}$ from the stability pattern for the case of an oblate body.

On the whole, the interaction of the effects of internal friction and external aerodynamic damping gives strange forms to the boundaries of the stability regions.

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